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Flux stabilization of D-branes

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Abstract: We explain how D-branes on group manifolds are stabilized against shrinking by quantized worldvolume U(1) fluxes. Starting from the Born-Infeld action in the case of the SU(2) group manifold we derive the masses, multiplicities and spectrum of small fluctuations of these branes, and show that they agree exactly with the predictions of Conformal Field Theory, to all orders in the α' expansion. We discuss the generalization to other groups and comment on an apparent paradox: why are the 'RR charges' of these branes not quantized?

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1. Introduction

String compactifications on group manifolds have long been of interest. Their world-sheet theories are the exactly solvable Wess-Zumino-Witten (WZW) conformal field theories and thus stringy effects can be understood in detail. The case of primary interest in superstring theory is the SU(2) group manifold, because the near-horizon geometry of k coincident NS fivebranes is a direct product including S^3, and the corresponding CFT is a product of supersymmetric SU(2) level k, a "Feigin-Fuchs superfield", and six free superfields [12]. Another exact supersymmetric string background [13] is $S^3 \times AdS_3$ — this corresponds to the CFT of two supersymmetric WZW models, one for the SU(2) and one for the $\mathrm{SL}(2,\mathbb{R})$ group manifold, and it describes the near-horizon geometry of intersecting branes [14, 15].

D-branes in group manifolds have been studied in a series of papers [2]–[11] and the basic story, at least for the compact case, is fairly well understood. The natural boundary conditions (those for which the gluing can be expressed in terms of an automorphism ω of the current algebra) can be classified purely in CFT terms. In the case of trivial gluing, $\omega = 1$, Cardy's general theory [1], puts them in one-to-one correspondence with primary fields.[1] The results turn out to be geometrical: [6, 7] an allowed boundary condition corresponds to a D-brane wrapped on an allowed (twisted) conjugacy class of the group. The only sign of the underlying CFT is a

[1] Cardy's theory can be generalized to non-trivial ω; in this case, one obtains a correspondence of boundary conditions with primary fields in twisted sectors of appropriate orbifold theories, [7].

quantization condition on the allowed (twisted) conjugacy classes. For example, in the SU(2) level k model D-branes can wrap on $k-1$ distinct S^2's around any point (these are subject to a $\mathbb{Z}_2$ identification). There is also a D0-brane, to complete the spectrum (there is no D3 because $H \neq 0$ [16]).

One then can study the world-volume theories of these branes by classifying massless modes and computing interactions. The results show an intriguing parallel with the noncommutative torus in that the algebra of open string primary fields (for large k but finite conjugacy class) is the algebra of the "fuzzy sphere", the natural quantization of the sphere [8].

However, there is a more elementary question one might ask first. Why is it that branes wrapped about spheres which are not minimal volume surfaces are stable at all?

It is easiest to check that the other boundary conditions lead to stable branes by considering the parallel with the boundary state describing a D2-brane ending on an NS fivebrane. This boundary state is a tensor product of "D0" in the WZW sector with "Neumann" in the linear dilaton sector and in two of the Minkowski dimensions. It preserves half of the supersymmetry of the fivebrane theory. If $k > 1$, the WZW component of the boundary state can be replaced with a different WZW boundary condition, leaving everything else unchanged. In particular the supersymmetry of the object is unchanged, so it is stable. The geometrical interpretation of these objects is "conical" D4-branes again wrapping a non-minimal S^2. Thus our question is appropriate.

It seems clear that this stability is linked with the origin of the quantization condition, and there are two ways one might try to explain this. One might imagine that the integrated NS two-form potential $\int \hat{B}$ takes quantized values on a D-brane world-sheet. This would imply an implausible non-local constraint on the allowed embeddings of the D-brane, with no known origin in string theory.

A better idea is that it follows from the usual quantization of U(1) gauge field strength, relevant because the D-brane is wrapped on S^2. Indeed there is a well known mechanism of "flux stabilization" (which has been invoked in large extra dimension scenarios, for example) which could then explain the brane's stability. It is simply that the energy $\int F^2$ of a constant flux will be inversely proportional to the volume, and thus the total energy with the brane tension will have a minimum at non-vanishing volume.

This argument is not really correct, because, unlike what happens in the Maxwell energy, the Born-Infeld energy of a constant flux stays finite as the brane shrinks to zero volume (this is why there are no stable spherical D-branes in flat spacetime). What enters in the D-brane energy, on the other hand, is the flux of the gauge-invariant combination $\mathcal{F} = B + 2\pi\alpha' F$. In a varying external B field the total energy including the brane tension can indeed have a minimum at nonzero volume, as we will show.

Our main result is to show that this explanation works not only qualitatively but quantitatively: indeed, up to the well known one loop shift $k \to k+2$ which renormalizes the radius of S^3, computations starting from the Born-Infeld action precisely reproduce the masses, multiplicities, and even the spectrum of small fluctuations of these branes, as calculated in CFT. The exact agreement implies that higher-order corrections to the Born-Infeld theory must vanish — this could be related to the BPS property of the corresponding objects in the fivebrane or $S^3 \times AdS_3$ geometries.

We consider the above results convincing evidence for the advocated explanation of stability. They do in particular confirm the fact that it is the U(1) flux $\int F$ (rather than the flux of $\mathcal{F}$) that must be quantized. This leads, however, to an apparent paradox: the RR charges of the branes are not quantized. We will explain why the F-flux quantization is correct, and discuss possible resolutions of the paradox in the final section.

2. Semiclassical brane solutions

Consider the WZW model on the group manifold of SU(2) — we will comment on the generalization to other groups later. A general group element can be parametrized as $U = \exp(i\vec{\psi} \cdot \vec{\sigma})$, where $\vec{\psi}$ is a 3-vector of length ψ pointing in the direction (θ, ϕ), and $\vec{\sigma}$ are the usual Pauli matrices. The coordinate ψ takes values in the interval $[0, \pi]$ with the two extremes corresponding to the two elements of the center. In these coordinates the metric and Neveu-Schwarz three-form backgrounds read

$$ds^2 = k\alpha' \left[d\psi^2 + \sin^2\psi \Big(d\theta^2 + \sin^2\theta \; d\phi^2 \Big) \right], \tag{2.1}$$

and

$$H \equiv dB = 2k\alpha' \sin^2\psi \sin\theta \, d\psi \, d\theta \, d\phi \,, \tag{2.2}$$

with k the (integer) level of the associated current algebra. We can choose a gauge in which the NS two-form is proportional to the volume form of the two-sphere spanned by (θ, ϕ),

$$B = k\alpha' \left(\psi - \frac{\sin 2\psi}{2} \right) \sin\theta \, d\theta \, d\phi \,. \tag{2.3}$$

This is a smooth choice everywhere except at the point $\psi = \pi$. The wavefunction of a fundamental string wrapping around this potential singularity picks up a Bohm-Aharonov phase equal to $\int_{S^2} B/2\pi\alpha' = 2\pi k$. The singularity is therefore unobservable for integer k as it should be.

Let us next put this WZW model together with seven flat space-time coordinates, so that the full geometry is $S^3 \times \mathbb{R}^7$. This is a non-critical background for type-II string theory because the central charges don't add up to ten, but the dilaton tadpole will not affect our discussion of D-branes at leading order in the string-loop expansion.

Consider now a static D2-brane wrapping the (θ, ϕ) two-sphere at fixed value of ψ. This configuration breaks the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry of the background to a diagonal SU(2). If the dominant brane energy were tensive our configuration would tend to shrink to a point at one of the two poles of S^3, either $\psi = 0$ or $\psi = \pi$. The total brane energy, on the other hand, has contributions also from the induced NS-NS two-form $\hat{B}$ and from the worldvolume gauge field $F = dA$, which enter through the invariant combination $\mathcal{F} = \hat{B} + 2\pi\alpha' F$. Consistently with the symmetry we may turn on a uniform worldvolume flux,

$$F = dA = -\frac{n}{2} \sin\theta \; d\theta d\phi \,, \tag{2.4}$$

where n is the 'magnetic monopole' number. For $0 < n < k$ one can check that $|\mathcal{F}|$ is locally maximum at the poles, so this could prevent the D2-brane from collapsing.

A crucial point in the further considerations is that it is the flux of F, rather than that of $\mathcal{F}$, which is quantized. This may seem counterintuitive in that F is not invariant under the gauge transformations $\delta\hat{B} = 2\pi\alpha' d\Lambda$ and $\delta A = -\Lambda$. One might have expected the quantization condition to apply to the gauge invariant $\mathcal{F}$.

A first comment one can make is that gauge transformations for which Λ is single-valued do not affect $\int F$, so claiming that $\int F$ is quantized is not evident nonsense. Although Λ need not be single-valued, in fact such large gauge transformations can only shift $\int F$ by an integer. This is how the usual quantization condition on $\int H$ arises in space-time language: one must define $\hat{B}$ in patches on S^3, and the allowed transition functions between the patches are those respecting this quantization condition.

This shows that the claim that $\int F$ and not $\int \mathcal{F}$ is quantized is sensible, but does not really prove it. Indeed, any argument for this claim which starts from a conventional world-volume gauge theory (such as the Born-Infeld action) would be circular, as the conventional gauge potential only makes sense if $\int F$ is quantized.

As is by now well-known, there do exist other gauge theories such as noncommutative gauge theory in which $\int F$ is not quantized in the usual way. However, the examples in which this is known to make sense at present are related to manifolds with non-trivial fundamental group, such as the torus. Indeed the case of S^2 has been much studied and the only noncommutative gauge theories which are known to make sense in this case are based on the "fuzzy sphere" [17] and have finitely many degrees of freedom (the original algebra of functions on S^2 is truncated). There is even a "no-go" theorem [18] to the effect that deformations of the algebra of functions which do not make this truncation, and which respect the natural SO(3) symmetry, do not exist as bounded algebras.

Although this is a theorem, we have not proven that its assumptions are the physically appropriate ones, and so we are not claiming at this point to show that noncommutative gauge theory with a field theoretic number of degrees of freedom does not exist on S^2. The point of this discussion is to explain why there is no sensible candidate low energy theory (at this writing) to describe the alternate hypothesis that $\int F$ is not quantized.

In any case, our hypothesis that $\int F$ is quantized will be confirmed shortly by the beautiful agreement of our results with those of conformal field theory.

Let us now fill in the appropriate formulae. The energy of our D2-brane with n units of worldvolume magnetic flux in the semiclassical (large-k) limit reads

$$\begin{aligned} E_n(\psi) &= T_{(2)} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{\det(\hat{G}+\mathcal{F})} + \cdots \\ &= 4\pi k\alpha' T_{(2)} \left(\sin^4\psi + \left(\psi - \frac{\sin 2\psi}{2} - \frac{\pi n}{k}\right)^2 \right)^{1/2} + \cdots \end{aligned} \tag{2.5}$$

where $T_{(2)}$ is the D-brane tension and the dots stand for higher α' corrections (see for instance [19]). For $0 < n < k$ this expression has a unique minimum away from the poles, at

$$\psi_n = \frac{\pi n}{k}\,, \tag{2.6}$$

where it takes the value

$$M_n = 4\pi k\alpha'\, T_{(2)} \sin\frac{\pi n}{k}\,. \tag{2.7}$$

For values of n outside this range the minimum of the energy is at $\psi = 0$ (if $n < o$) or $\psi = \pi$ (if $n > k$) and it corresponds to a singular configuration of the brane. This gives a total of $k-1$ non-singular configurations. In order to take into account the well-known one loop shift of the sphere curvature, we should replace everywhere k by $k+2$. There is however no reason at this point to trust our expressions beyond the large k and n (with n/k held fixed) limit, since only in this limit are the wordvolume curvatures small in string units.

We can also evaluate the D-particle charge induced by the background flux on the above stable non-degenerate D2-branes. Using the standard formulae [20] one finds

$$Q_n = T_{(2)} \int_{S^2} \mathcal{F} \;=\; 2\pi k\alpha'\, T_{(2)} \sin\frac{2\pi n}{k}\,. \tag{2.8}$$

These charges are not even rationally related to each other — here is the apparent paradox we have alluded to in the introduction. Note also that in the flat limit, $k \to \infty$ with n held fixed, eqs. (2.7) and (2.8) reduce to the mass and charge of n free D-particles, $E_n \simeq Q_n \simeq nT_{(0)}$. The above stable configurations should have in fact a dual description as bound states of n D-particles on the sphere, which can be analyzed along the lines of [21]. From the exact properties of these bound states we can place restrictions on the non-abelian Born-Infeld theory, similar to those of [22, 23], but this is outside the scope of the present work.

3. Small fluctuations

The 'mini-superspace' analysis of the previous section took only into account the degree of freedom corresponding to rigid motions of the D2-brane in the ψ-direction. In this section we derive the complete spectrum of small quadratic fluctuations around the above D-brane solutions. This will allow us to confirm their stability, to find their classical moduli space, and to compare later on with the spectrum of boundary operators for the corresponding Cardy states.

We use static gauge in which the worldvolume is parametrised by (t, θ, ϕ), and impose $A_0 = 0$ for the worldvolume gauge field. We ignore for simplicity brane fluctuations and gauge-field components in the extra spectator spatial directions, and concentrate on the three remaining degrees of freedom

$$\psi = \frac{\pi n}{k} + \delta\,, \quad A_\theta = \frac{k}{2\pi}\alpha_\theta\,, \qquad \text{and} \qquad A_\phi = \frac{n}{2}(\cos\theta - 1) + \frac{k}{2\pi}\alpha_\phi\,. \tag{3.1}$$

Here the small fluctuations $\delta, \alpha_\theta, \alpha_\phi$ are arbitrary functions on $R \times S^2$, and the $k/2\pi$ normalization is introduced for convenience. The Born-Infeld energy-density reads

$$\mathcal{L}_{BI} = T_{(2)}\sqrt{-\det\,(\hat{G} + \mathcal{F})}\,, \tag{3.2}$$

where

$$\hat{G} + \mathcal{F} = k\alpha' \begin{pmatrix} -\frac{1}{k\alpha'} + (\partial_t\delta)^2 & \partial_t\delta\partial_\theta\delta + \partial_t\alpha_\theta & \partial_t\delta\partial_\phi\delta + \partial_t\alpha_\phi \\ \partial_t\delta\partial_\theta\delta - \partial_t\alpha_\theta & \sin^2\psi + (\partial_\theta\delta)^2 & \partial_\theta\delta\partial_\phi\delta + \mathcal{F}_{\theta\phi} \\ \partial_t\delta\partial_\phi\delta - \partial_t\alpha_\phi & \partial_\theta\delta\partial_\phi\delta - \mathcal{F}_{\theta\phi} & \sin^2\psi\sin^2\theta + (\partial_\phi\delta)^2 \end{pmatrix}, \tag{3.3}$$

with

$$\mathcal{F}_{\theta\phi} \equiv \left(\delta - \frac{\sin(2\psi)}{2}\right)\sin\theta + \partial_\theta\alpha_\phi - \partial_\phi\alpha_\theta\,. \tag{3.4}$$

In expanding out the determinant to quadratic order, terms involving δ off the diagonal drop out. After some tedious but straightforward algebra the Born-Infeld lagrangian up to quadratic order takes the form

$$\begin{aligned}\mathcal{L}_{BI} \propto\; & 2f\cot\left(\frac{n\pi}{k}\right) + k\alpha'\sin\theta\left[(\partial_t\delta)^2 + (\partial_t\alpha_\theta)^2 + \frac{(\partial_t\alpha_\phi)^2}{\sin^2\theta}\right] - \\ & - \left(\frac{1}{\sin\theta}\right)\left[(\partial_\phi\delta)^2 + \sin^2\theta(\partial_\theta\delta)^2 + 2\sin^2\theta\delta^2 + 4\sin\theta\delta f + f^2\right]. \end{aligned}\tag{3.5}$$

We have here denoted $f \equiv \partial_\theta\alpha_\phi - \partial_\phi\alpha_\theta$ for short, and have dropped the leading, fluctuation-independent term of the lagrangian as well as an irrelevant multiplicative constant.

The above expression starts out with a linear term, which seems to contradict our assertion that we are expanding around a classical solution. This is however not the case. The linear term is proportional to the fluctuation of the integrated flux, which must be set to zero because of the quantization condition. This demonstrates explicitly that it is the magnetic flux that stabilizes the D-brane. The quadratic terms in (3.5) are furthermore independent of the background solution we expand around. This means that the spectrum of fluctuations is independent of n, in agreement with the conformal field-theory result as we will see soon.

From the above lagrangian we derive the following linearized equations for the fluctuation fields,

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta\\ \alpha_\theta\\ \alpha_\phi\end{pmatrix} = \mathcal{O}\begin{pmatrix}\delta\\ \alpha_\theta\\ \alpha_\phi\end{pmatrix}, \tag{3.6}$$

where the operator-valued matrix is

$$\mathcal{O} = -\frac{1}{k\alpha'}\begin{pmatrix}\Box + 2 & -\frac{2}{\sin\theta}\partial_\phi & \frac{2}{\sin\theta}\partial_\theta\\ \frac{2}{\sin\theta}\partial_\phi & -\frac{1}{\sin^2\theta}\partial_\phi^2 & \frac{1}{\sin^2\theta}\partial_\phi\partial_\theta\\ -\frac{2}{\sin\theta}\partial_\theta & \sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\phi & -\sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\theta\end{pmatrix} \tag{3.7}$$

and

$$\Box = -\frac{1}{\sin^2\theta}\partial_\phi^2 - \frac{1}{\sin\theta}\,\partial_\theta\sin\theta\,\partial_\theta \tag{3.8}$$

is the covariant laplacian on S^2. The operator $\mathcal{O}$ has zero eigenvalues corresponding to the unphysical longitudinal polarization of the photon. We can extract the transverse polarization by combining the last two equations so as to express everything in terms of the physical fluctuation f. After some algebra the answer is

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta\\ f/\sin\theta\end{pmatrix} = -\frac{1}{k\alpha'}\begin{pmatrix}\Box + 2 & 2\\ 2\Box & \Box\end{pmatrix}\begin{pmatrix}\delta\\ f/\sin\theta\end{pmatrix}. \tag{3.9}$$

We can now readily diagonalize this operator by going to a basis of spherical harmonics,

$$\delta = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\delta_{lm}Y_{lm}(\theta,\phi) \qquad \text{and} \qquad f = \sin\theta\sum_{l=1}^{\infty}\sum_{m=-l}^{l}f_{lm}Y_{lm}(\theta,\phi) \tag{3.10}$$

with the reality conditions $\delta_{lm} = \delta^*_{l\ -m}$ and similarly for f. Notice that the absence of the s-wave in the expansion of f guarantees the flux quantization condition, as can be checked using the orthonormality of the spherical harmonics. For $l = 0$ there is therefore only the δ-fluctuation, and its frequency squared is $2/k\alpha'$. For all other l we need to consider the matrix

$$\frac{1}{k\alpha'}\begin{pmatrix}l(l+1)+2 & 2\\ 2l(l+1) & l(l+1)\end{pmatrix}, \tag{3.11}$$

whose eigenvalues are $(l+1)(l+2)/k\alpha'$ and $l(l-1)/k\alpha'$. Putting it all together we thus have the following spectrum of quadratic fluctuations

$$m^2 = j(j+1)/k\alpha', \quad \text{in reps.} \quad (j-1)\oplus(j+1), \qquad \text{for } j = 0, 1, 2, \ldots \tag{3.12}$$

with the understanding that only the spin-one representation appears in the special case $j = 0$. This corresponds precisely to a triplet of zero modes, corresponding to arbitrary rotations of the worldvolume two-sphere inside S^3. All other excitations have positive mass, confirming the stability of our solutions.

4. CFT analysis

Let us now compare the results of the last two sections with those of conformal field theory. In the diagonal-invariant bosonic theory there are $(k+1)$ Cardy [1] boundary states preserving a SU(2) symmetry,

$$|q\rangle\!\rangle_C \equiv \sum_{p=0}^{k} \frac{S_{qp}}{\sqrt{S_{0p}}}\, |p\rangle\!\rangle_I\,, \tag{4.1}$$

where $|p\rangle\!\rangle_I$ is the Ishibashi (character) state corresponding to the chiral primary of spin $p/2$, and

$$S_{ij} = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right) \tag{4.2}$$

is the modular-transformation matrix.

We want to identify these states with the semiclassical configurations of section 2. We have seen that there are $k-1$ stable non-degenerate D2-branes, when the S^3 radius is $\sqrt{k\alpha'}$, and adding the two D-particles at the north and south poles gives the correct total number. Alternatively, if we take into account the quantum shift $k \to k+2$, we find exactly $k+1$ non-degenerate D2 branes. These two ways of counting are of course indistinguishable in the semiclassical large k limit, since it is hard to tell the difference between a point-like D-particle and one with radius $\sim \sqrt{\alpha'}$ (this is the radius of the worldvolume for $n = 1$ units of flux). Adopting the latter point of view makes, however, the Born-Infeld results exact — this may be related to supersymmetry in the fivebrane context, but we did not have any reason to expect it a priori. To exhibit this precise agreement of the formulae we will assume that the S^3 radius is $\sqrt{(k+2)\alpha'}$, and identify

$$|n-1\rangle\!\rangle_C \quad \leftrightarrow \quad (\text{flux-}n\ \text{D2-brane})\,, \tag{4.3}$$

where the flux n takes the values $1, 2, \ldots, k+1$.

The mass of a Cardy state can be read off from the coefficient of its $p = 0$ Ishibashi component which has non-vanishing overlap with the (seven-dimensional) graviton and dilaton, see for instance [24]. To be more precise, the interaction energy between two D-branes a distance r apart due to the exchange of the (seven-dimensional) graviton and dilaton can be calculated along the lines of [20] with the result[2]

$$\mathcal{E}(r) = 2\kappa_{(7)}^2\, M_n^2\, \Delta_{(6)}(r)\,, \tag{4.4}$$

[2] We can formally ignore the dilaton tadpole in this calculation.

where $\kappa_{(7)}$ is the gravitational coupling in seven dimensions, M_n the mass of the D-branes and $\Delta_{(6)}(r)$ the six-dimensional euclidean Green's function. The string-theory calculation for this interaction energy on the other hand is

$$\mathcal{E}(r) = (2\pi)^4 \frac{|S_{n0}|^2}{S_{00}} \, \Delta_{(6)}(r)\,, \tag{4.5}$$

where we have here projected onto the identity-operator in the closed-string channel of the amplitude, and we have set $\alpha' = 1/2$. Comparing the two expressions gives

$$M_n = \frac{2\sqrt{2}\pi^2}{\kappa_{(7)}} \, \frac{|S_{n0}|}{\sqrt{S_{00}}} \; = \; \frac{(2\pi)^{3/2}(2k+4)^{1/4}}{\kappa_{(7)}} \sin\left(\frac{n\pi}{k+2}\right), \tag{4.6}$$

where we have here used the expressions for the S-matrix coefficients. Using now the relations

$$T_{(2)} = \frac{\sqrt{2\pi^3}}{\kappa_{(10)}}, \qquad \text{and} \qquad \kappa^2_{(10)} = \kappa^2_{(7)} \left(\frac{k+2}{2}\right)^{3/2} 2\pi^2 \tag{4.7}$$

one can verify that the above mass agrees precisely with our semiclassical result (2.7), including all the numerical prefactors.

Consider next the spectrum of quadratic fluctuations, to be compared with the open-string excitations in the $\mathcal{H}_{(n-1)(n-1)}$ Hilbert space. If we neglect transverse spatial dimensions, the light states in this Hilbert space are of the form

$$|\text{open}\rangle = J^a_{-1}|j\rangle \tag{4.8}$$

where J^a are the SU(2) currents and $|j>$ is created by a primary field with $j = 0, \ldots, k/2$. These transform in the $(j-1)\oplus j \oplus (j+1)$ representations of SU(2), but imposing the (super)Virasoro constraint will project the representation j out of the spectrum. One way to see this is to note that there are as many constraints as number of primaries, namely $2j+1$, and since physical states must form SU(2) representations it is necessarily the j representation that is projected out. The conformal weight of the vertex operators corresponding to the states (4.8) is $j(j+1)/(k+2)$, in agreement with the semiclassical mass formula of small fluctuations derived in section three.

There is one thing that the semiclassical analysis misses: this is the cutoff on the allowed spins j of the open strings living on a brane with n units of flux. In the CFT only the representations with spin less than $\min(n, k/2-n)$ enter in the annulus amplitude. In the Dirac-Born-Infeld analysis on the other hand all spins j are a priori allowed, so this stringy effect is not visible. This should be contrasted to the cutoff on the allowed U(1) flux ($n < k+2$) which is correctly reproduced by the semiclassical analysis.

Another qualitative confirmation of the results in section two follows from an analysis of the 'wavefunctions' of the Cardy states in position space [7]. These are peaked around equally-spaced values of the polar angle ψ, in agreement again with the semiclassical result (2.6). Notice that the moduli space for rigid translations of the D2-branes on the group manifold corresponds to the freedom of obtaining equivalent Cardy states by group conjugation.

Finally, let us compare the induced D-particle charge (2.8) with the result of CFT. In the CFT this charge is given by the $p = 1$ coefficient of the Cardy state, because the corresponding closed-string RR states transform in the $(p/2 \otimes 1/2, p/2 \otimes 1/2)$ representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The reason is that the zero modes of the supersymmetric WZW fermions are realized on a bispinor of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. Now the D-particle charge of interest is a SO(4) singlet — this can be verified explicitly by checking that it does not transform under rigid translations of the D2-brane on S^3. Thus only $p = 1$ contributes to this coupling, and a calculation similar to the one for the mass gives

$$Q_n = \frac{(2\pi)^{3/2}(2k+4)^{1/4}}{2\kappa_{(7)}} \sin\left(\frac{2n\pi}{k+2}\right), \tag{4.9}$$

in perfect agreement again with the result of section two. That these charges are not rationally related to each other is thus confirmed by the CFT analysis — we will return to the point in the final section.

5. General group manifolds

In this section we discuss some aspects of the generalization of our results to compact Lie groups G which we assume to be simple, connected and, for simplicity, to be simply connected. Our discussion will be entirely topological, the precise form of the metric and antisymmetric tensor will not play any role. The reader not interested in this generalization can go directly to the final section.

The D-brane world volumes for all boundary conditions for which the gluing of left movers and right movers at the boundary is given by an automorphism ω of G have been described in [7]. They are (regular) twined conjugacy classes, i.e. they are subspaces of the form

$$\mathcal{C}_\omega(g) = \{hg\omega(h)^{-1} \text{ with } h \in G\} \tag{5.1}$$

where $g \in G$ is a regular element.

Let us describe the geometry of twined conjugacy classes in somewhat more detail: for any automorphism ω, there is a maximal torus T of G that is invariant under ω. The subgroup of elements of T that are left pointwise fixed by ω,

$$T^\omega = \{t \in T | \omega(t) = t\}, \tag{5.2}$$

is not a torus, but a semi-direct product of a torus and a finite abelian group; its connected component T_0^ω of the identity is a torus. In case ω is an inner automorphism — and this is always true for $G = \mathrm{SU}(2)$ — all subgroups of G coincide:

$$T = T_0^\omega = T^\omega\,. \tag{5.3}$$

For inner automorphisms, this torus actually coincides with a maximal torus and the dimension of the torus equals the rank of G; so e.g. for inner automorphisms of SU(3) we have a two-dimensional torus. For outer automorphisms, the dimension of T_0^ω is smaller than the rank; for outer automorphisms of SU(3) it is equal to one.

Weyl's classical theory of conjugacy classes has in fact a nice generalisation to twined conjugacy classes. We will sketch some statements of this theory. The central tool is the following: given a maximal torus T, we define a map from the coset space G/T and the maximal torus T to the group by using conjugation:

$$\begin{aligned} q:\ & G/T \times T \to G \\ & q([g], t) = gtg^{-1} \end{aligned} \tag{5.4}$$

Weyl could show that the mapping degree of q equals the number of elements in the Weyl group W. Maps with positive degree are surjective, so in particular one sees that any element of G is conjugated to some element in the maximal torus T. So any conjugacy class can be characterized by an element of the maximal torus. In fact, different elements of the maximal torus parametrize identical conjugacy classes if and only if they are related by the action of the Weyl group. In the case of SU(2), a maximal torus is one-dimensional; examples are given by circles of constant values of ϕ and θ; they can be parametrized by ψ. The Weyl group is just Z_2, and its action has been taken into account by restricting ψ to the range $0 \le \psi \le \pi$. Finally, fixing t we see that regular conjugacy classes are isomorphic to the homogeneous space G/T. In the case of SU(2) this gives SU(2)/ U(1) which is isomorphic to the two-sphere.

The results nicely generalize to twisted conjugacy classes (for details see [7]). For any automorphism ω, q is replaced by q_ω that is defined via twisted conjugation:

$$\begin{aligned} q_\omega:\ & G/T_0^\omega \times T_0^\omega \to G \\ & q_\omega([g], t) = gt\omega(g^{-1})\,. \end{aligned} \tag{5.5}$$

The mapping degree of q_ω can be shown to be positive. To state more precise results, we need the subgroup W_ω of the Weyl group W that commutes with the action of W on the weight space:

$$W_\omega := \{w \in W | \omega^* w = w\omega^* \text{ for all } w \in W\} \tag{5.6}$$

The group W_ω has been shown in [25] to be isomorphic to the Weyl group of the so-called orbit Lie algebra [26]. For the outer automorphism of SU(3) this group is Z_2, which is the Weyl group of the orbit Lie algebra SU(2).

The mapping degree of q_ω is just $n_{T^\omega}|W_\omega|$, where n_{T^ω} is the number of connected components of T^ω. Weyls classical results can now be generalized to twined conjugacy classes. All statements remain true, provided one replaces the maximal torus T by T_0^ω and the Weyl group W by W_ω: Twined conjugacy classes are characterized by elements of T_0^ω; different elements of T_0^ω describe identical twined conjugacy classes if and only if they are related by the action of W_ω. Regular twined conjugacy classes are isomorphic to the homogeneous space G/T_0^ω. Even a generalization of Weyl's integration formula holds [27].

To give an explicit example, regular D-branes for inner automorphisms of $\mathrm{SU}(3)$ are isomorphic to $\mathrm{SU}(3)/\mathrm{U}(1)^2$ and are thus six-dimensional. They are characterized by two parameters. For *outer* automorphisms, they are isomorphic to $\mathrm{SU}(3)/\mathrm{U}(1)$ and therefore seven-dimensional. For their characterization a single parameter suffices. Outer automorphisms therefore change the dimensionality of the worldvolume.

Extending the analysis of $\mathrm{SU}(2)$ to other groups requires a detailed knowledge of the differential geometry of the group manifold, in particular a good choice of coordinates. The corresponding calculations become rather complicated and are beyond the scope of the present note. However, there is a simple and yet non-trivial check of the stabilization mechanism: we expect as many independent $\mathrm{U}(1)$ fluxes as the number of transverse brane coordinates that must be stabilized.

The possible $\mathrm{U}(1)$ fluxes on the worldvolume of the D-brane are given by

$$\mathrm{dim} H^2(G/T_0^\omega, \mathrm{R})\,. \tag{5.7}$$

Following our previous discussion, the description of a specific D-brane requires on the other hand $\dim T_0^\omega$ parameters. We should thus expect the general relation

$$\dim H^2(G/T_0^\omega, \mathbf{R}) = \dim T_0^\omega\,. \tag{5.8}$$

Such a relation does indeed hold in full generality: for a simply connected compact Lie group G also the second homotopy group $\pi_2(G)$ vanishes. The long exact sequence in homotopy

$$\cdots \to \pi_k(G) \to \pi_k(G/T_0^\omega) \to \pi_{k-1}(T) \to \pi_{k-1}(G) \to \cdots \tag{5.9}$$

implies for $k = 1$ that the homogeneous space G/T_0^ω is simply connected and for $k = 2$ that $\pi_2(G/T_0^\omega)$ is isomorphic to $\pi_1(T_0^\omega)$. The latter is a free abelian group whose rank is $\dim T_0^\omega$. The homotopy group $\pi_2(G/T_0^\omega)$ therefore coincides with the homology we want to determine, and we find indeed that

$$H^2(G/T_0^\omega, R) \cong \pi_2(G/T_0^\omega) \cong \pi_1(T_0^\omega) \cong \mathbb{Z}^{\dim T_0^\omega} \tag{5.10}$$

Notice in particular that the line bundles over G/T_0^ω do not have any continuous parameters. This generalizes the situation of $\mathrm{SU}(2)$, where we consider bundles over S^2. This is reflected in the conformal field theory analysis by the fact that we find D-brane worldvolumes whose only continuous deformations are given by (inner) automorphisms of the group, but which do not have any other moduli.

The fact that the relation (5.8) always holds shows that the advocated mechanism could indeed be responsible for the stability of all known WZW D-branes.

6. Fivebrane and a paradox

To further discuss the physics of the SU(2) branes, let us consider a configuration of N coincident supersymmetric (NS) five-branes in type II theory.

The full fivebrane background is (in string frame)

$$\begin{aligned} ds^2 &= dx^2 + f(r)dy^2 \\ e^{2\phi} &= g_s^2 f(r) \\ f(r) &= 1 + \frac{N\alpha'}{r^2} \\ H &= N\alpha'\epsilon_3 \end{aligned} \tag{6.1}$$

where x are the $5+1$ longitudinal coordinates, y are 4 transverse coordinates and $r = |y|$.

In the near-horizon limit $r \to 0$, the background factorizes into a radial component and an S^3. The corresponding CFT has a Feigin-Fuchs (linear dilaton) field in addition to the supersymmetric SU(2) WZW model [12]. The supersymmetric WZW model can be realized as a tensor product of three free fermions (with level 2 current algebra) and a bosonic WZW model of level k (the k of the previous sections). The fivebrane number N is identified with the total central charge $k+2$ of the SU(2) current algebra. This may sound unsatisfactory, as there is apparently no candidate theory for a single five-brane, but it agrees with the standard lore that the center-of-mass degrees of freedom of the branes are not be visible in the dual holographic theory. A single fivebrane has no degrees of freedom other than center-of-mass, and hence no dual holographic description.

The D particles of the previous section are nonsupersymmetric and unstable in this background. They are momentum modes in the eleventh dimension, which tend to fall towards the core of the fivebrane where the eleventh dimension blows up and the D-particles become massless. This agrees with the expectation that they should complete SO(5) representations of the fivebrane fields [28].

D2-branes which end on the fivebrane are supersymmetric and correspond to the product of a Neumann boundary state in the linear dilaton theory, and a Dirichlet (D0) boundary state in the WZW model. In this context, the additional WZW boundary states will correspond to D4 branes extending along the radial direction (and so ending on the fivebranes), but now (in the near horizon regime) "wrapped" on an transverse S^2, forming a conical geometry.

To study the supersymmetry properties of these branes, we need to write down the space-time supersymmetry generators. The world-volume supersymmetry generators are given in [12] and the D2 boundary conditions in the WZW model preserve an $N = 4$ world-sheet supersymmetry. This is enough to guarantee that the world-volume operator corresponding to the space-time supercharge also exists with these boundary conditions.

The conjugate brane (in the sense of electric-magnetic duality) would be a D4-brane with three dimensions wrapped on S^3 and $1+1$ extending in Minkowski space. Although this of course does not exist because it is wrapping a surface with $H \neq 0$, there is a similar object which is believed to exist [28]. The total flux $\int H$ on the brane can be made zero by allowing k D2-branes to end on the brane (and extend outward), analogous to the "baryon" of [16].

The existence of the conjugate object would appear to require D2 charge quantization. So why is RR charge not quantized?

We first note that there is a superficially similar effect, already visible in toroidal compactification, with a much simpler explanation. Since the integrals $\int \hat{B}$ over two-cycles in the target space can take arbitrary non-zero values, the induced RR charges $\int C \wedge \hat{B}$ are non-integral. However, this is not a violation of quantization but rather a rotation of the entire charge lattice.

A simple way to see that this is not what is happening here is to realize that charge quantization requires that there be an integral basis of the charge lattice of rank equal to the number of charges (one must then check that the DSZ form is integral of course). In the present case (and for SU(2)), we have found $O(k/2)$ distinct 'charges' satisfying no integer relations, but at most two independent RR charges (from the two-dimensional cohomology of S^3).

The way one avoids an immediate contradiction is by noting that the RR fields in question are massive in the near-horizon geometry. This can be seen in the CFT where the normalizable vertex operators have (six-dimensional) mass bounded below by the background charge of the Feigin-Fuchs coordinate [28]. Alternatively, in the low energy effective theory, this follows from the Chern-Simons coupling $\int G \wedge G \wedge B$. We work in the usual string conventions in which the RR kinetic terms and sources are independent of the dilaton. This leads to

$$d * G = H \wedge G + \delta^{(7)} + B \wedge \delta^{(5)} \tag{6.2}$$

and

$$dG = \delta^{(5)} \tag{6.3}$$

where $G = dC^{(3)}$ is the four-form field strength, $\delta^{(9-p)}$ is the source associated to a $p+1$-brane (a $9-p$ form normal to the world-volume). The conserved electric and magnetic RR charges are then

$$Q_M = \int G$$

and

$$Q_E = \int *G - B \wedge G.$$

In the near-horizon limit of the five-brane background, B is independent of distance r. In this case the CS term makes the RR field effectively massive, so the quantization condition will not be visible.

In the true five-brane background, when we go to asymptotic infinity (the Minkowski region), the B field does fall off with distance, and the charge quantization must become observable again. This regime is of course not described by our explicit CFT. Since the volume of the S^3 grows with radius in the normal way here, presumably the conical four-brane must asymptote to a cylindrical four-brane (or even n two-branes again). This fits with the asymptotic BPS bound which implies that the tension here is just n times that of the original two-brane.

In a general background with non-zero H, it is clear that the induced RR charge will depend on the embedding of the brane through $\int \hat{B}$ and so this contribution cannot be quantized. However this variation could be cancelled by the bulk CS term: the total variation of the right hand side of (6.2) under a variation of the embedding of the D4-brane is zero, under a suitable interpretation of the boundary terms.[3]

These points seem to us to resolve the paradox both in the context of the near-horizon geometry (because the RR fields are massive) and in the full geometry (in which the H flux falls off fast enough to apply the second argument). However they leave open the interesting question of just what the CFT results (4.9) are measuring. One possibility is that they are related to the $N-1 = k+1$ independent charges of the holographically dual field theory on the fivebranes. For N type IIB fivebranes the worldvolume theory has $\mathrm{SU}(N)$ gauge symmetry, and hence $N-1$ independent charges. These charges may indeed correspond to the different allowed couplings of the massive RR field in the bulk theory.

We feel there is more to say about this issue, but will leave it for future work.

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[3] W. Taylor [29] has recently shown that taking this effect into account indeed leads to quantization of the total charge, as suggested by our discussion. The question raised in the final paragraph remains open.

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